

On the Logarithmic Coefficients of Close to Convex Functions

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Abstract

For f analytic and close to convex in $D = \{z : |z| < 1\}$, we give sharp estimates for the logarithmic coefficients γ_n of f defined by $\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$ when $n = 1, 2, 3$.

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Introduction

Let S be the class of normalised analytic univalent functions f for $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

The logarithmic coefficients of f are defined in D by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \quad (1)$$

The logarithmic coefficients γ_n play a central role in the theory of univalent functions. Milin conjectured that for $f \in S$ and $n \geq 2$,

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$$\sum_{m=1}^n \sum_{k=1}^m (k|\gamma_k|^2 - \frac{1}{k}) \leq 0 \quad (2)$$

and it is not difficult to see that (2) implies the Bieberbach conjecture. It was a proof of (2) that De Branges established in order to prove the conjecture.

Very few exact upper bounds for γ_n seem have been established, with more attention being given to results of an average sense (see e.g. [1, 2]). Moreover it is known that for $f \in S$, the expected inequality $|\gamma_n| \leq \frac{1}{n}$ is false even in order of magnitude [1, Theorem 8.4].

Differentiating (1) and equating coefficients gives

$$\gamma_1 = \frac{1}{2}a_2 \quad (3)$$

$$\gamma_2 = \frac{1}{2}(a_3 - \frac{1}{2}a_2^2) \quad (4)$$

$$\gamma_3 = \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3) \quad (5)$$

Hence $|\gamma_1| \leq 1$ follows at once from (3), and use of the Fekete-Szegő inequality in (4), [1, Theorem 3.8] gives the sharp estimate

$$|\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635\dots$$

For $n \geq 3$, the problem seems much harder, and no significant upper bounds for $|\gamma_n|$ when $f \in S$ appear to be known.

Denote by S^* the subclass of S of starlike functions, so that $f \in S^*$ if, and only if, for $z \in D$

$$Re \frac{zf'(z)}{f(z)} > 0.$$

Thus we can write $zf'(z) = f(z)h(z)$, where $h \in P$, the class of functions satisfying $Re h(z) > 0$ for $z \in D$. Simple differentiation in (1) again and

noting that the coefficients c_n of the Taylor series of h about $z = 0$ satisfy $|c_n| \leq 2$ for $n \geq 1$, shows that $|\gamma_n| \leq \frac{1}{n}$ holds for $f \in S^*$ and $n \geq 2$.

Suppose now that f is analytic in D , then f is close-to-convex if, and only if, for $z \in D$, there exists $g \in S^*$ such that

$$Re \frac{zf'(z)}{g(z)} > 0. \quad (6)$$

We denote the class of close to convex functions by K and note the well-known inclusion relationship $S^* \subset K \subset S$.

That the inequality $|\gamma_n| \leq \frac{1}{n}$ for $n \geq 2$ extends to the class K was claimed in a paper of El Hosh [3]. However Girela [4] pointed out an error in the proof and showed that for $f \in K$, this inequality is false for $n \geq 2$. In the same paper it was shown that $|\gamma_n| \leq \frac{3}{2n}$ holds for $n \geq 1$ whenever f belongs to the set of the extreme points of the closed convex hull of the class K , which implies that $|\gamma_3| \leq \frac{1}{2}$ in this case. As was pointed out above, this bound false for the entire class K . It is the purpose of this paper to establish the sharp bound $|\gamma_3| \leq \frac{7}{12}$ for the class K when the coefficient b_2 in the Taylor expansion for $g(z)$ is real.

We first note that from (4) it is an immediate consequence of the Fekete-Szegő inequality for $f \in K$ [5] that the following sharp inequality holds for $f \in K$

$$|\gamma_2| = \frac{1}{2}|a_3 - \frac{1}{2}a_2^2| \leq \frac{11}{18} = 0.6111..$$

We now turn our attention to the case $n = 3$ for the class K .

It follows from (6) that we can write $zf'(z) = g(z)h(z)$, where $Re h(z) > 0$ for $z \in D$ and, since $g \in S^*$, $zg'(z) = g(z)p(z)$, where $Re p(z) > 0$ for $z \in D$.

Now write

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (7)$$

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (8)$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \quad (9)$$

We shall need the following result [6], which has been used widely.

Lemma

Let $h, p \in P$ and be given by (7) and (8) respectively, then for some complex valued x with $|x| \leq 1$ and some complex valued t with $|t| \leq 1$

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t. \end{aligned}$$

Similarly for some complex valued y with $|y| \leq 1$ and some complex valued s with $|s| \leq 1$

$$\begin{aligned} 2p_2 &= p_1^2 + y(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1y - p_1(4 - p_1^2)y^2 + 2(4 - p_1^2)(1 - |y|^2)s. \end{aligned}$$

We prove the following:

Theorem

Let $f \in K$, then

$$|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{11}{18}.$$

Also when $f \in K$ and b_2 is real,

$$|\gamma_3| \leq \frac{7}{12}.$$

The inequalities are sharp.

Proof. As noted above, the first two inequalities are proved. Thus it remains to prove the third.

From (5) we need to find an upper bound for

$$|\gamma_3| = \frac{1}{2}|a_4 - a_2a_3 + \frac{1}{3}a_2^3|. \quad (10)$$

First note that equating coefficients we have

$$\begin{aligned} 2a_2 &= c_1 + p_1 \\ 3a_3 &= c_2 + c_1p_1 + \frac{p_1^2 + p_2}{2} \\ 4a_4 &= c_3 + c_2p_1 + \frac{c_1(p_1^2 + p_2)}{2} + \frac{p_1^3}{6} + \frac{p_1p_2}{2} + \frac{p_3}{3}. \end{aligned}$$

Substituting into (10) gives

$$\begin{aligned} |a_4 - a_2a_3 + \frac{1}{3}a_2^3| &= \left| \frac{c_3}{4} + \frac{c_2p_1}{12} + \frac{c_1p_2}{24} + \frac{p_3}{12} + \frac{p_1p_2}{24} \right. \\ &\quad \left. - \frac{c_1c_2}{6} - \frac{c_1^2p_1}{24} + \frac{c_1^3}{24} \right|. \end{aligned} \quad (11)$$

We now use the Lemma to eliminate c_2 , c_3 , p_2 and p_3 from (11) and obtain

$$\begin{aligned} |a_4 - a_2a_3 + \frac{1}{3}a_2^3| &= \left| \frac{c_1^3}{48} + \frac{c_1xX}{24} - \frac{c_1x^2X}{16} + \frac{XZ}{8} + \frac{p_1xX}{24} + \frac{c_1p_1^2}{48} \right. \\ &\quad \left. + \frac{c_1yY}{48} + \frac{p_1yY}{16} - \frac{p_1y^2Y}{48} + \frac{p_1^3}{24} + \frac{YV}{24} \right| \end{aligned} \quad (12)$$

where for simplicity, we have set $X = 4 - c_1^2$, $Y = 4 - p_1^2$, $Z = (1 - |x|^2)s$ and $V = (1 - |y|^2)t$.

Without loss in generality we may write $c_1 = c$, with $0 \leq c \leq 2$. Also since we are assuming $b_2 = p_1$ to be real, we can write $p_1 = q$, with $0 \leq |q| \leq 2$. Writing $|q| = p$, it then follows using the triangle inequality in (12) together with $|s| \leq 1$ and $|t| \leq 1$, that

$$\begin{aligned}
|a_4 - a_2a_3 + \frac{1}{3}a_2^3| &\leq \frac{c^3}{48} + \frac{c|x|X}{24} + \frac{c|x|^2X}{16} + \frac{XZ}{8} + \frac{p|x|X}{24} + \frac{cp^2}{48} \\
&\quad + \frac{c|y|Y}{48} + \frac{p|y|Y}{16} + \frac{p|y|^2Y}{48} + \frac{p^3}{24} + \frac{YV}{24} \\
&= F(c, p, |x|, |y|) \text{ say.}
\end{aligned} \tag{13}$$

where now $X = 4 - c^2$, $Y = 4 - p^2$, $Z = 1 - |x|^2$ and $V = 1 - |y|^2$.

Thus we need to find the maximum of $F(c, p, |x|, |y|)$ over the hyper-rectangle $R = [0, 2] \times [0, 2] \times [0, 1] \times [0, 1]$.

From (13) substituting for X , Y , Z and V gives

$$\begin{aligned}
F(c, p, |x|, |y|) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}c|x|(4 - c^2) + \frac{1}{16}c|x|^2(4 - c^2) \\
&\quad + \frac{1}{8}(4 - c^2)(1 - |x|^2) + \frac{1}{24}p|x|(4 - c^2) + \frac{1}{48}c|y|(4 - p^2) \\
&\quad + \frac{1}{16}p|y|(4 - p^2) + \frac{1}{48}p|y|^2(4 - p^2) + \frac{1}{24}(4 - p^2)(1 - |y|^2).
\end{aligned} \tag{14}$$

We first assume that $F(c, p, |x|, |y|)$ has a maximum value at an interior point $(c_0, p_0, |x_0|, |y_0|)$ of R . Then since

$$\frac{\partial F}{\partial |x|} = \frac{1}{24}c(4 - c^2) + \frac{1}{4}c|x|(4 - c^2) - \frac{1}{4}|x|(4 - c^2) + \frac{1}{2}p(4 - c^2) = 0$$

at such a point, it follows that $c_0 = 2$, which is a contradiction. Hence any maximum points must be on the boundary of R .

Thus we need to find the maximum value of $F(c, p, |x|, |y|)$ on each of the 32 edges and 24 faces (8 of co-dimension 1 and 16 of co-dimension 2) of R . Finding these maximum values involves a great many tedious exercises in elementary calculus and for the sake of brevity, we omit many of the simple ones. The process does however identify the maximum value of $7/6$ needed in the Theorem and shows that the maximum value on all edges and faces is less than or equal to $7/6$.

Finding the maximum values of $F(c, p, |x|, |y|)$ on each of the 32 edges involves trivial exercises, and shows that $F(c, p, |x|, |y|) \leq 7/6$ on all of these edges. On the 16 faces of co-dimension 2, similar simple exercises in elementary calculus again shows that $F(c, p, |x|, |y|) \leq 7/6$ on each of these faces. We thus consider the 8 faces of co-dimension 1 as follows.

On the face $c = 0$, suppose that $|x| \leq 1$ in (14), which gives a resulting expression

$$\begin{aligned} G_1(0, p, |y|) &= \frac{1}{24}p^3 + \frac{1}{2} + \frac{1}{6}p + \frac{1}{16}p|y|(4 - p^2) \\ &\quad + \frac{1}{48}p|y|^2(4 - p^2) + \frac{1}{24}(4 - p^2)(1 - |y|^2). \end{aligned}$$

Differentiating $G_1(0, p, |y|)$ with respect to $|y|$ shows that any maximum must occur on the boundary of $[0, 2] \times [0, 1]$ and since the largest value at the end points is $7/6$, $F(c, p, |x|, |y|)$ has maximum $7/6$ on the face $c = 0$.

On the face $c = 2$, suppose again that $|x| \leq 1$ in (14), to obtain the expression

$$\begin{aligned} G_2(2, p, |y|) &= \frac{1}{6} + \frac{1}{24}p^2 + \frac{1}{24}p^3 + \frac{1}{24}|y|(4 - p^2) \\ &\quad + \frac{1}{16}p|y|(4 - p^2) + \frac{1}{48}p|y|^2(4 - p^2) + \frac{1}{24}(4 - p^2)(1 - |y|^2), \end{aligned}$$

and following the same procedure gives a maximum of 0.696 on $[0, 2] \times [0, 1]$.

On the face $p = 0$, suppose that $|x| \leq 1$ and $|y| \leq 1$ in (14), to obtain the expression

$$G_3(c, 0, |y|) = \frac{1}{48}c^3 + \frac{5}{48}c(4 - c^2) + \frac{1}{8}(4 - c^2) + \frac{1}{12}c + \frac{1}{6},$$

which has maximum value $23/24$ on $[0, 2] \times [0, 1]$.

On the face $p = 2$, (14) becomes

$$G_4(c, 2, |x|) = \frac{1}{3} + \frac{c}{12} + \frac{1}{48}c^3 + \frac{1}{12}(4 - c^2)|x| + \frac{1}{24}c(4 - c^2)|x| \\ + \frac{1}{16}c(4 - c^2) + \frac{1}{8}(4 - c^2)(1 - |x|^2).$$

Differentiating $G_4(c, 2, |x|)$ with respect to $|x|$ and considering the end points gives a maximum value 1.005 on $[0, 2] \times [0, 1]$.

On the face $|x| = 0$, suppose that $|y| \leq 1$ in (14), to obtain

$$G_5(c, p, 0) = \frac{1}{48}c^3 + \frac{1}{8}(4 - c^2) + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}(4 - p^2) \\ + \frac{1}{12}p(4 - p^2) + \frac{1}{48}c(4 - p^2),$$

and it is now an easy exercise to show that $G_5(c, p, 0)$ has a maximum value of 0.9531 when $p = 4/3$ and $c = 2 - 2\sqrt{6}/3$ on $[0, 2] \times [0, 2]$.

On the face $|x| = 1$, suppose that $|y| \leq 1$ in (14), to obtain

$$G_6(c, p, 1) = \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{5}{48}c(4 - c^2) + \frac{1}{24}p(4 - c^2) \\ + \frac{1}{24}(4 - p^2) + \frac{1}{12}p(4 - p^2) + \frac{1}{48}cp(4 - p^2) \\ = \frac{1}{6} + \frac{1}{2}c - \frac{1}{12}c^3 + \frac{1}{2}p - \frac{1}{24}c^2p - \frac{1}{24}p^2 - \frac{1}{24}p^3 \\ \leq \frac{1}{6} + \frac{1}{2}c - \frac{1}{12}c^3 + \frac{1}{2}p - \frac{1}{24}p^3.$$

It is now a simple exercise to show that this expression has maximum value $5/6$ on $[0, 2] \times [0, 2]$

On the face $|y| = 0$, (14) becomes

$$G_7(c, p, |x|) = \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}(4 - p^2) + \\ + \frac{1}{24}c|x|(4 - c^2) + \frac{1}{24}p|x|(4 - c^2) + \frac{1}{16}c|x|^2(4 - c^2) \\ + \frac{1}{8}(4 - c^2)(1 - |x|^2).$$

Differentiating $G_7(c, p, |x|)$ with respect to $|x|$ shows as before, that there are no maximum points in the interior of $[0, 2] \times [0, 2] \times [0, 1]$, and so we need only find the maximum values of $G_7(c, p, |x|)$ on the boundary of $[0, 2] \times [0, 2] \times [0, 1]$. In the interests of brevity, we omit the simple analysis which gives maximum value of 1.005 when $p = 2$ and $|x| = 1$

We finally note that on the face $|y| = 1$

$$\begin{aligned} G_8(c, p, |x|) &= \frac{1}{48}c^3 + \frac{1}{48}cp^2 + \frac{1}{24}p^3 + \frac{1}{24}c|x|(4 - c^2) + \frac{1}{16}c|x|^2(4 - c^2) \\ &\quad + \frac{1}{8}(4 - c^2)(1 - |x|^2) + \frac{1}{24}p|x|(4 - c^2) + \frac{1}{48}c(4 - p^2) \\ &\quad + \frac{1}{12}p(4 - p^2). \end{aligned}$$

As before, differentiating $G_8(c, p, |x|)$ with respect to $|x|$ shows that there are no maximum points in the interior of $[0, 2] \times [0, 2] \times [0, 1]$, and so we need only find the maximum values of $G_8(c, p, |x|)$ on the boundary of $[0, 2] \times [0, 2] \times [0, 1]$. Again in the interests of brevity, we omit the simple analysis which gives a maximum value of 1.052, again less than $7/6$.

Thus we have shown that in all cases, the maximum value of (14) is at most $7/6$, which completes the proof of the Theorem.

We finally note that equality in the inequality in $|\gamma_3| \leq 7/12$ is attained when $c_1 = 0$ and $c_2 = c_3 = p_1 = p_2 = p_3 = 2$.

□

Remark 1

The condition that b_2 is real in the inequality for $|\gamma_3|$ arises in order to maximise (12). We conjecture that this condition can be removed and $|\gamma_3| \leq 7/12$ for $f \in K$.

Remark 2

The correct growth rate for γ_n appears to be unknown for close-to-convex functions and in this direction the best known estimate to date appears to be that of Ye [7], who showed that $|\gamma_n| \leq \frac{A \log n}{n}$, where A is an absolute constant.

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